Topology of $SU(N)/\mathbb{Z}_N$ lattice gauge fields and generalized 't Hooft anomaly

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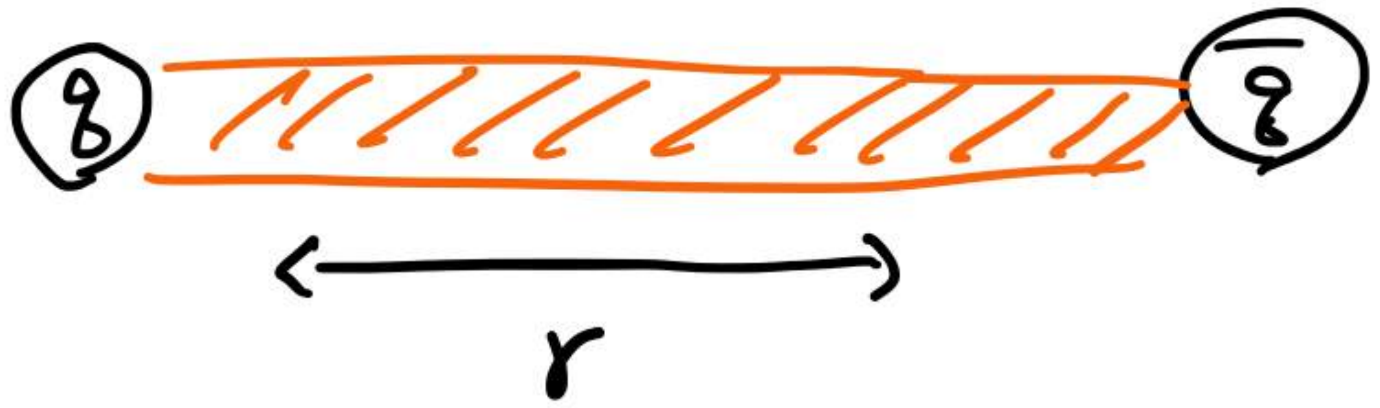
4d $SU(N)$ Yang-Mills theory

- is a fundamental theory of strong interaction,
- generates a mass gap

$$\Lambda \sim \frac{1}{a} e^{-\frac{8\pi^2}{\beta_0 g^2}} \qquad \left(\beta_0 = \frac{11}{3}N\right),$$

- shows confinement of color-electric charges.



$$V(r) \sim \sigma \cdot r .$$
$$(\sigma \sim \Lambda^2)$$

In these days, it has been uncovered that the confining vacua of YM theories have richer structures than we've expected.

4d $SU(N)$ gauge fields have topological sectors

$$\frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F) \in \mathbb{Z},$$

so the YM theory has the θ-angle:

$$Z_\theta = \int \mathcal{D}a \; \exp\left(-\frac{1}{g^2}\int \mathrm{tr}(F\wedge * F) + i\,\underbrace{\frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)}\right).$$

The θ-angle is periodic because

$$Z_{\theta+2\pi} = Z_\theta .$$

However, the vacua at $\theta=0$ and $\theta=2\pi$ are distinct !!

Introduce $\mathbb{Z}_N$ 2-form background gauge field B, then

$$Z_{\theta+2\pi}[B] = \underbrace{e^{i\frac{N}{4\pi}\int B\wedge B}}_{} \; Z_\theta[B].$$

Local counterterm is shifted. (Gaiotto, Kapustin, Komargodski, Seiberg 2017)

This key relation $\boxed{Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B^2} Z_\theta[B]}$ — (*)

was derived by analyzing the smooth and classical gauge fields.

<u>Question</u> Does the relation (*) hold for the renormalized quantum YM theory?

<u>Our answer</u> By extending the work of Lüscher (1982),

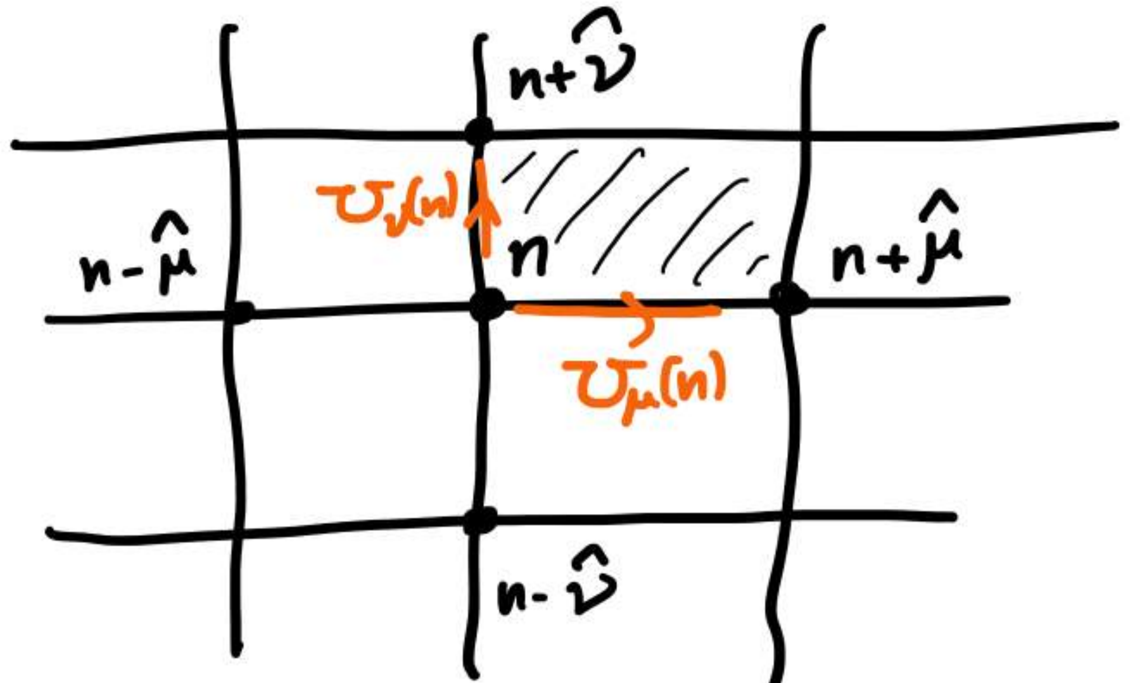
we can define the admissibility for lattice $SU(N)$ gauge fields coupled with B, and the admissible lattice gauge fields have $Q_{top}[U_\ell, B_p]$ s.t.

- Q_{top} is (ultra-) local,
- Q_{top} is ($SU(N)$- and $\mathbb{Z}_N$ 1-form -) gauge invariant, and
- $Q_{top} = \underbrace{\frac{N}{4\pi}\int B\wedge B}_{\in \frac{1}{N}\mathbb{Z}} + \text{(integer)}$.

In short, $\boxed{\text{the relation (*) is true at the finite lattice spacings.}}$

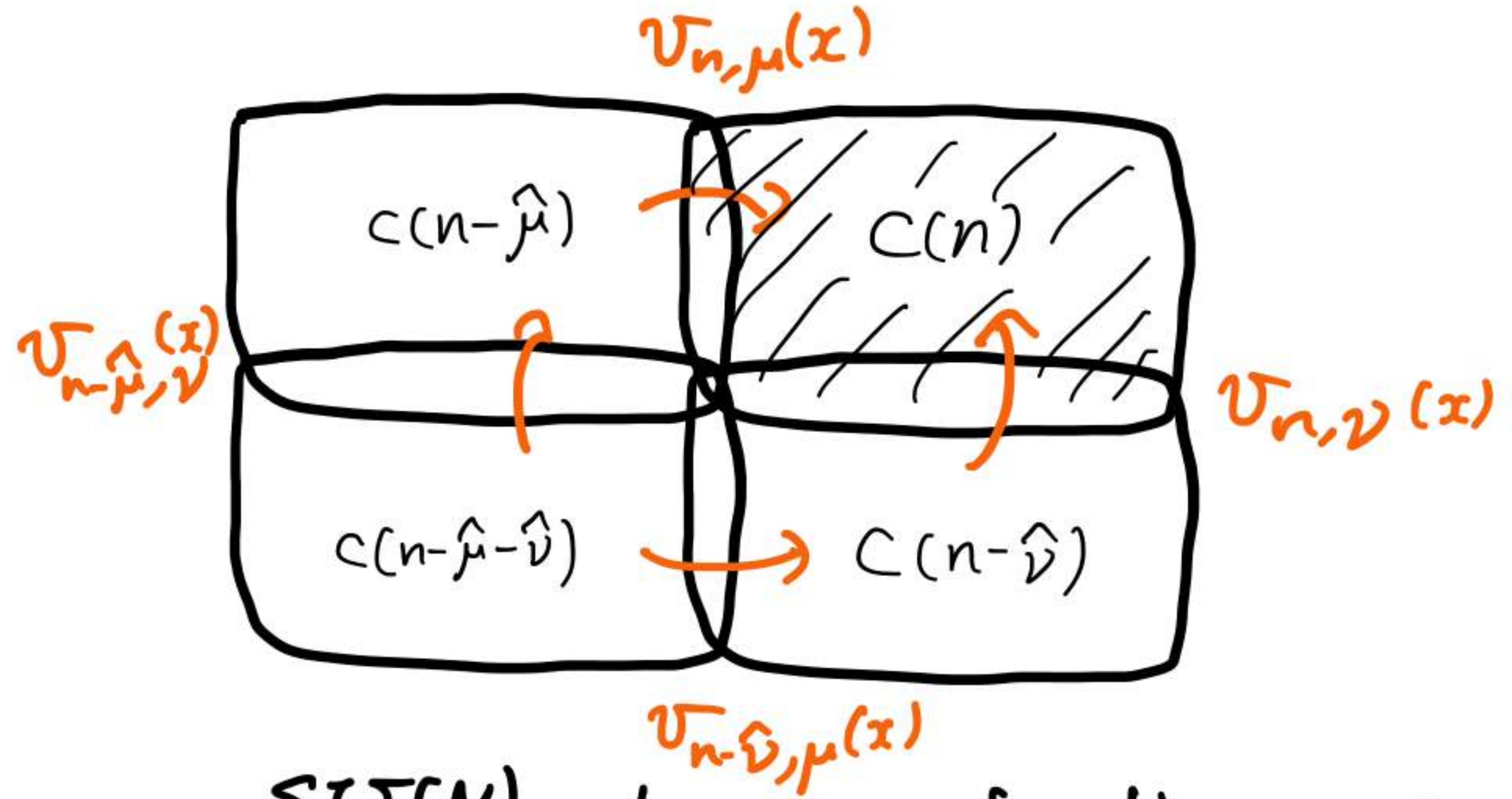
Idea of Lüscher (1982)

Topological charge $Q_{top} \sim \int \mathrm{tr}(F \wedge F)$ depends only on the transition functions.



SU(N) lattice gauge fields $U_\mu(n)$

$\Rightarrow$



SU(N) transition functions s.t.

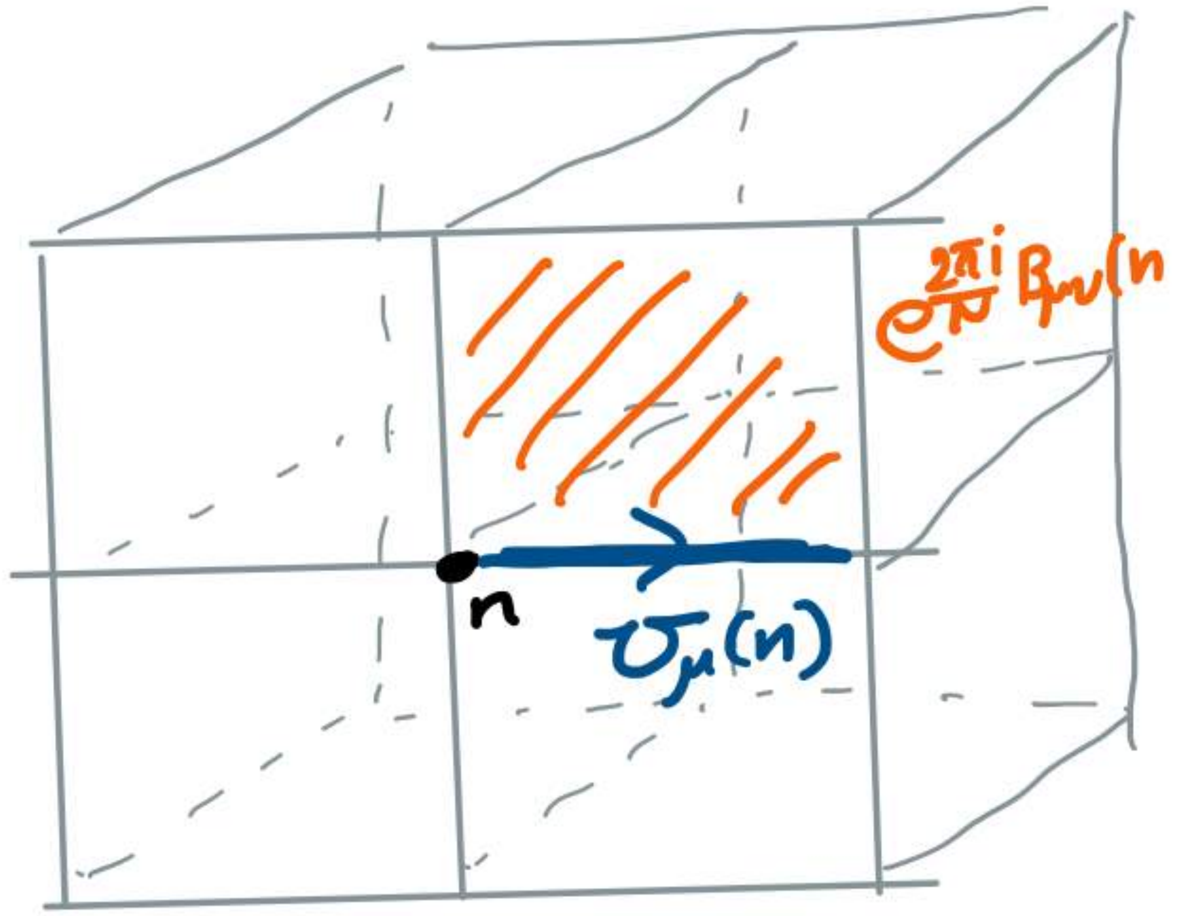
$$v_{n,\mu}(x)\, v_{n-\hat\mu,\nu}(x) = v_{n,\nu}(x)\, v_{n-\hat\nu,\mu}(x) \quad \text{for } x \in c(n) \cap c(n-\hat\mu) \cap c(n-\hat\nu) \cap c(n-\hat\mu-\hat\nu).$$

Constructing such $v_{n,\mu}(x)$ out of $\{U_\mu(n)\}$, we can define

$$Q_{top} \in \mathbb{Z}.$$

This is possible if $\| 1 - \underset{\overset{\shortparallel}{U_\mu(n) U_\nu(n+\hat\mu) U^\dagger_\mu(n+\hat\nu) U^\dagger_\nu(n)}}{U_{\mu\nu}(n)} \| < \varepsilon\ (\simeq 0.1)$ ("admissibility condition")

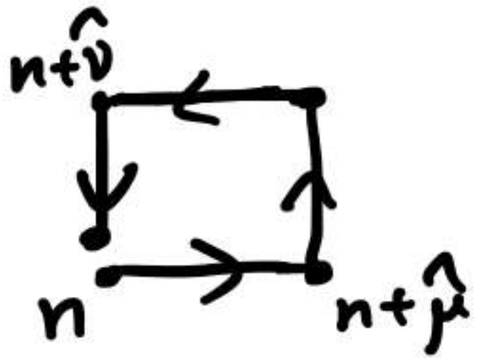
Lattice $SU(N)$ gauge fields w/ $\mathbb{Z}_N$ 2-form gauge fields.



$$\begin{cases} U_\mu(n) \in SU(N) & : SU(N) \text{ link variable} \\ e^{\frac{2\pi i}{N}B_{\mu\nu}(n)} \in \mathbb{Z}_N & : \mathbb{Z}_N \text{ plaquette variable} \end{cases}$$

Wilson gauge action :

$$S_W[U_\mu(n), B_{\mu\nu}(n)] = \beta \sum_{n,\mu,\nu} \left\{ \mathrm{tr}\left(1 - \underbrace{e^{-\frac{2\pi i}{N}B_{\mu\nu}(n)} U_{\mu\nu}(n)}_{=\tilde{U}_{\mu\nu}(n)} \right) + \text{c.c.} \right\}$$

w/ $U_{\mu\nu}(n) =$  $= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu(n+\hat{\nu})^{-1}\, U_\nu(n)^{-1}$.

- $\{B_{\mu\nu}(n)\}$ is assumed to be flat :

$$(\Delta B)_{\mu\nu\rho}(n) := \Delta_\mu B_{\nu\rho}(n) - \Delta_\nu B_{\mu\rho}(n) + \Delta_\rho B_{\mu\nu}(n) = 0 \quad (\mathrm{mod}\ N).$$

↖ lattice exterior derivative

- S_W is invariant under

$$\begin{cases} \bullet\ SU(N) \text{ gauge tranoformation}, & U_\mu(n) \to g(n)^\dagger U_\mu(n)\, \underset{}{\overset{SU(N)}{g}}(n+\hat{\mu}), \text{ and} \\ \bullet\ \mathbb{Z}_N \text{ 1-form gauge transformation}, & U_\mu(n) \to e^{\frac{2\pi i}{N}\lambda_\mu(n)} U_\mu(n),\ B_{\mu\nu}(n) \to B_{\mu\nu}(n) + (\Delta\lambda)_{\mu\nu}(n). \end{cases}$$

Admissibility condition (Lüscher, 1982)

- The space of lattice gauge fields $\{U_\mu(n)\}$ is connected.

  (Pf Any config. $\{U_\mu(n)\}$ can be continuously deformed to the trivial config. $\{\mathbb{1}\}_{n,\mu}$.)

  $\Rightarrow$ Unlike the continuum case, there are no topological sectors.

- Note that the continuum limit is the weak coupling limit: $\Lambda \sim \frac{1}{a} e^{-\frac{(8\pi^2/\frac{11}{3}N)}{g^2}}$.

  The path integral is dominated by

$$\| \mathbb{1} - \tilde{U}_{\mu\nu}(n) \| \lesssim \mathcal{O}(\sqrt{g^2}).$$

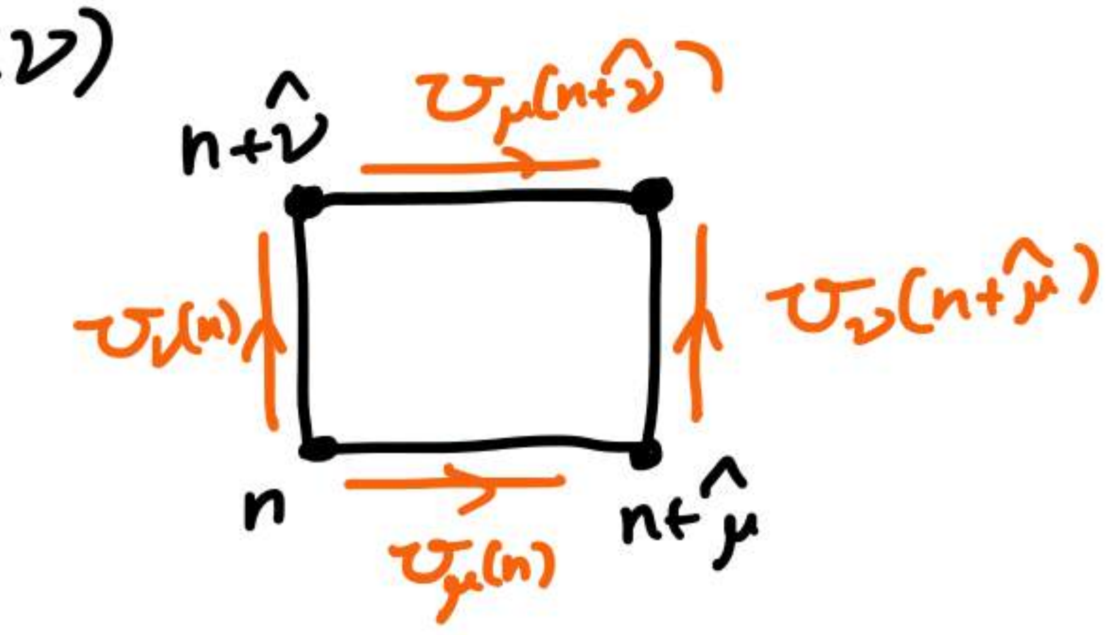
$\Rightarrow$ Pick some ε ($\simeq 0.1$), and restrict the path integral to the "admissible" fields

$$\underbrace{\| \mathbb{1} - \tilde{U}_{\mu\nu}(n) \|}_{\text{Local \& gauge invariant}} < \underbrace{\varepsilon}_{\text{Just a number independent from physical parameters, such as } g^2\text{, lattice size, } \cdots}.$$

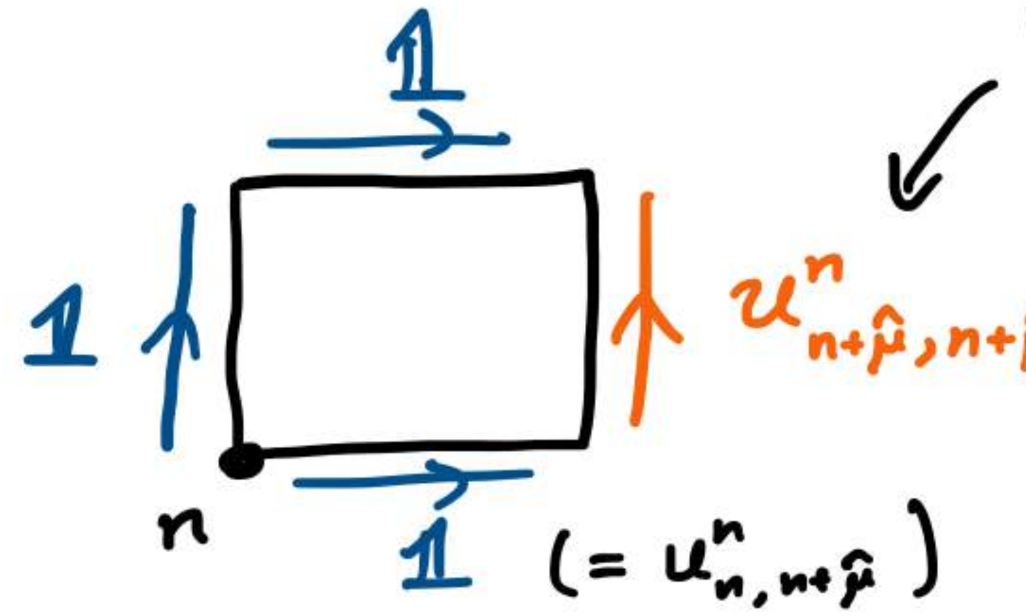
Sketch for the construction of transition functions $\tilde{v}_{n,\mu}(n)$

- On each cell, we take the complete axial gauge

$(\mu<\nu)$



$\Longrightarrow$



link variable for the axial gauge "at n".

$u^n_{n+\hat{\mu},n+\hat{\mu}+\hat{\nu}} = \underbrace{v_\mu(n)}\, v_\nu(n+\hat{\mu})\, \underbrace{(v_\nu(n)\, v_\mu(n+\hat{\nu}))^{-1}}$

"standard parallel transporter"

$\Longrightarrow$ Attach $B_{\mu\nu}$'s to u^n_{ab} so that they are $\mathbb{Z}_N$ 1-form gauge invariant:

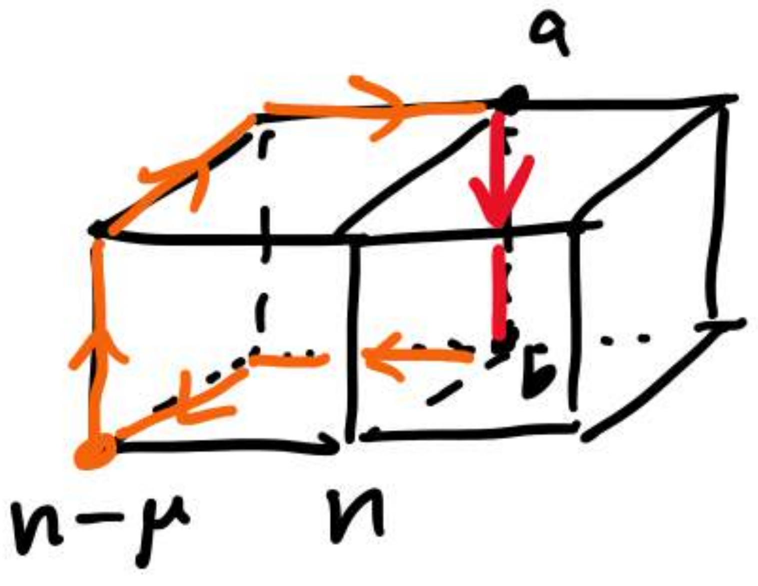
$$\tilde{u}^n_{ab} = e^{\frac{2\pi i}{N}(B+\cdots)}\, u^n_{ab}.$$

- Define transition functions $\tilde{v}_{n,\mu}(x)$ at the corner of $f(n,\mu) = c(n-\hat{\mu}) \cap c(n)$.

For the link $\langle a \to b\rangle \in f(n,\mu)$, two axial gauges at n and at $n-\hat{\mu}$ are defined:

$$\tilde{u}^{n-\hat{\mu}}_{ab} = \underline{\tilde{v}_{n,\mu}(a)}\; \tilde{u}^n_{ab}\; \underline{\tilde{v}_{n,\mu}(b)}^{-1}.$$

transition functions.



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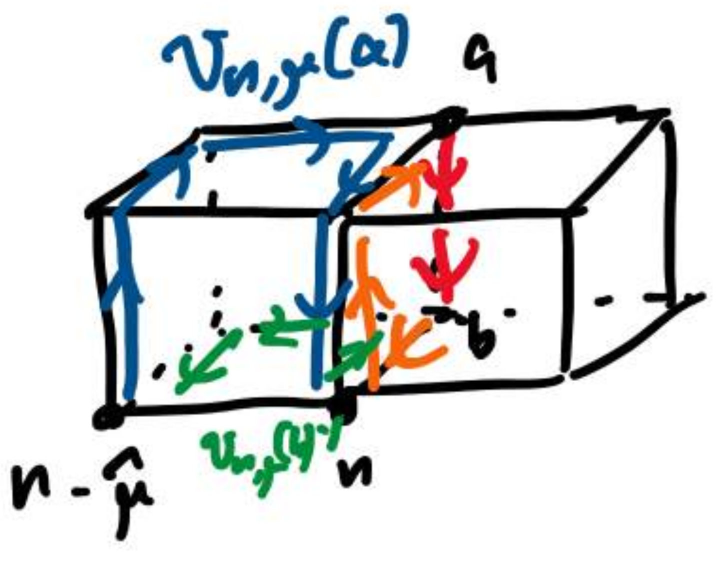


Interpolation of $\tilde{v}_{n,\mu}$ for admissible gauge fields

We define $\tilde{v}_{n,\mu} : f(n,\mu) \to SU(N)$ by

transition func. at the corner is already defined

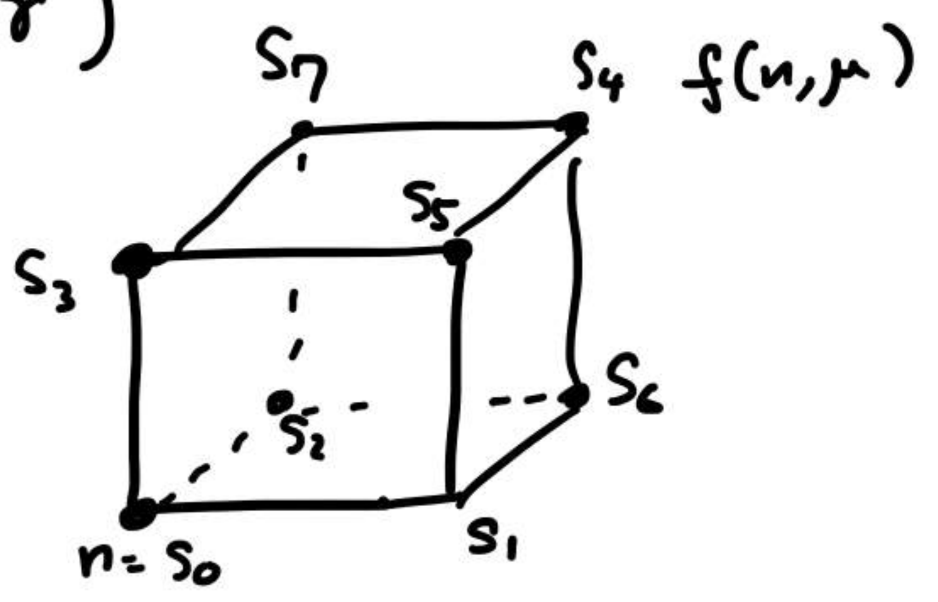
$$\tilde{v}_{n,\mu}(x) := \boxed{\tilde{S}^{\,n-\hat{\mu}}_{n,\mu}(x)^{-1}}\; \tilde{v}_{n,\mu}(n)\; \boxed{\tilde{S}^{\,n}_{n,\mu}(x)}$$

smooth interpolating functions.

so that

$$\tilde{v}_{n-\hat{\nu},\mu}(x)\, \tilde{v}_{n,\nu}(x)\, \tilde{v}_{n,\mu}(x)^{-1}\, \tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\, \mathbb{1} \quad \text{on } x \in \overbrace{p(n,\mu,\nu)}^{c(n)\cap c(n-\hat{\mu})\cap c(n-\hat{\nu}).}$$

Such interpolating function can be defined as $(x = n + y_\alpha \hat{\alpha} + y_\beta \hat{\beta} + y_\gamma \hat{\gamma})$



$$\tilde{f}^m_{n,\mu}(x_\gamma) = (\tilde{u}^m_{s_3 s_0})^{y_\gamma} (\tilde{u}^m_{s_0 s_3} \tilde{u}^m_{s_3 s_7} \tilde{u}^m_{s_7 s_2} \tilde{u}^m_{s_2 s_0})^{y_\gamma} \tilde{u}^m_{s_0 s_2} (\tilde{u}^m_{s_2 s_7})^{y_\gamma},$$

$$\tilde{g}^m_{n,\mu}(x_\gamma) = (\tilde{u}^m_{s_5 s_1})^{y_\gamma} (\tilde{u}^m_{s_1 s_5} \tilde{u}^m_{s_5 s_4} \tilde{u}^m_{s_4 s_6} \tilde{u}^m_{s_6 s_1})^{y_\gamma} \tilde{u}^m_{s_1 s_6} (\tilde{u}^m_{s_6 s_4})^{y_\gamma},$$

$$\tilde{h}^m_{n,\mu}(x_\gamma) = (\tilde{u}^m_{s_3 s_0})^{y_\gamma} (\tilde{u}^m_{s_0 s_3} \tilde{u}^m_{s_3 s_5} \tilde{u}^m_{s_5 s_1} \tilde{u}^m_{s_1 s_0})^{y_\gamma} \tilde{u}^m_{s_0 s_1} (\tilde{u}^m_{s_1 s_5})^{y_\gamma},$$

$$\tilde{k}^m_{n,\mu}(x_\gamma) = (\tilde{u}^m_{s_7 s_2})^{y_\gamma} (\tilde{u}^m_{s_2 s_7} \tilde{u}^m_{s_7 s_4} \tilde{u}^m_{s_4 s_6} \tilde{u}^m_{s_6 s_2})^{y_\gamma} \tilde{u}^m_{s_2 s_6} (\tilde{u}^m_{s_6 s_4})^{y_\gamma},$$

$$\tilde{l}^m_{n,\mu}(x_\beta, x_\gamma) = \left[\tilde{f}^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \left[\tilde{f}^m_{n,\mu}(x_\gamma) \tilde{k}^m_{n,\mu}(x_\gamma) \tilde{g}^m_{n,\mu}(x_\gamma)^{-1} \tilde{h}^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \cdot \tilde{h}^m_{n,\mu}(x_\gamma) \left[\tilde{g}^m_{n,\mu}(x_\gamma)\right]^{y_\beta},$$

$$\boxed{\tilde{S}^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma)} = (\tilde{u}^m_{s_0 s_3})^{y_\gamma} \left[\tilde{f}^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \left[\tilde{l}^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}.$$

Topological charge on the lattice

Transition functions $\{\tilde{v}_{n,\mu}\}$ define the $SU(N)/\mathbb{Z}_N$ principal bundle.

$$\Rightarrow \quad Q_{top}[U_\ell, B_p] = \sum_n q(n),$$

with
$$q(n) = \frac{1}{24\pi^2}\sum_\mu (-1)^\mu \int_{f(n,\mu)} \mathrm{tr}\left[(\tilde{v}_{n,\mu}^{-1} d\tilde{v}_{n,\mu})^3\right]$$
$$+ \frac{1}{8\pi^2}\sum_{\mu,\nu} (-1)^{\mu+\nu} \int_{p(n,\mu,\nu)} \mathrm{tr}\left[(\tilde{v}_{n,\mu} d\tilde{v}_{n,\mu}^{-1}) \wedge (\tilde{v}_{n-\hat{\mu},\nu}^{-1} d\tilde{v}_{n-\hat{\mu},\nu})\right].$$

This topological charge is defined in a local, gauge-invariant way.

It enjoys the quantization condition:

$$\boxed{Q_{top}[U_\ell, B_p] = -\frac{1}{N}\int_{T^4} \frac{1}{2}(B \cup B + B \cup_1 \delta B) \quad + \quad (\text{integers}).}$$

$$\Rightarrow \quad Z_{\theta+2\pi}[B] = e^{-\frac{2\pi i}{N}\int_{T^4}\frac{1}{2}P_2(B)} Z_\theta[B].$$

Summary

- Lattice $SU(N)$ gauge fields w/ admissibility have the well-defined topological charge. (Lüscher, 1982)
  local, gauge-inv.

- We can extend this feature to the case w/ $\mathbb{Z}_N$ 2-form background fields.

- This gives the rigorous lattice derivation of the 't Hooft anomaly

$$Z_{\theta+2\pi}[B] = e^{\frac{2\pi i}{N}\int \frac{1}{2}P_2(B)} Z_\theta[B].$$

Future direction

- Can we extend our result to the overlap fermions?
  $\Rightarrow$ QCD w/ adjoint quarks.

- Can we couple periodic axions to this topological charge?